EE.351: Spectrum Analysis and Discrete-Time Systems
FINAL EXAM, 9:00AM–12:00PM, December 12, 2003 **(closed book)**

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Note: **There are six questions. All questions are of equal value but not necessarily of equal difficulty. Full marks shall only be given to solutions that are properly explained and justified.**

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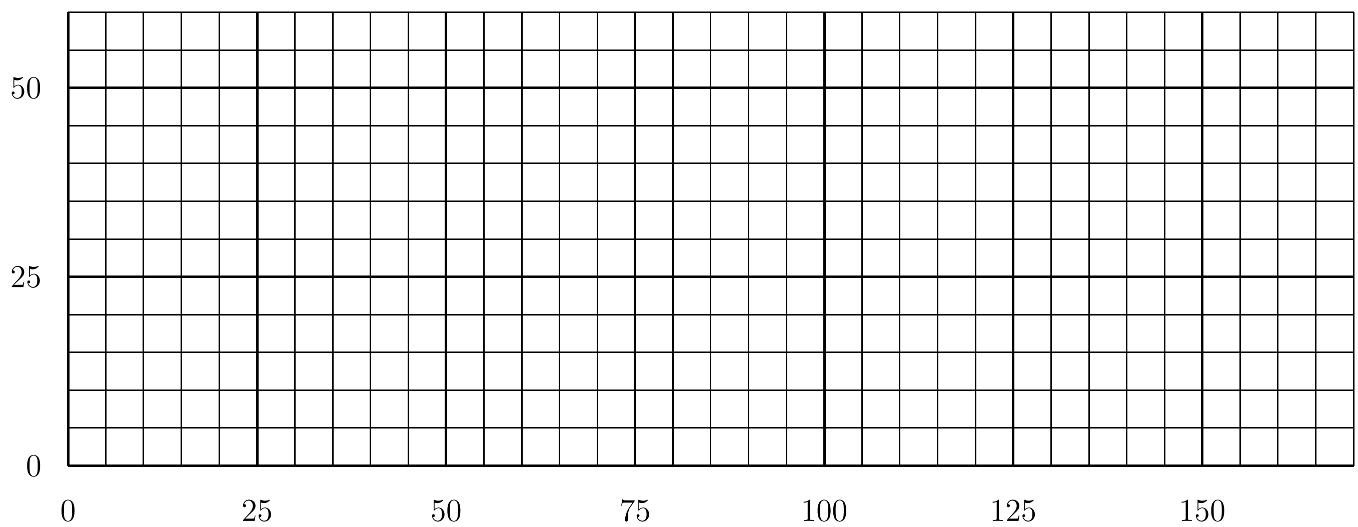
1. *(FS of a continuous-time signal)* Consider a half-wave rectifier whose output $y(t)$ is related to the input $x(t)$ as follows: $y(t) = \begin{cases} x(t), & \text{if } x(t) \geq 0 \\ 0, & \text{if } x(t) < 0 \end{cases}$. Let $x(t) = \cos(2\pi t)$.

[3] (a) Neatly sketch the input $x(t)$ and the output $y(t)$. What are the fundamental periods of $x(t)$ and $y(t)$?



[5] (b) Determine the Fourier series coefficients of the output $y(t)$.

[2] (c) What are the amplitudes of the dc components of the input and output signals, respectively?

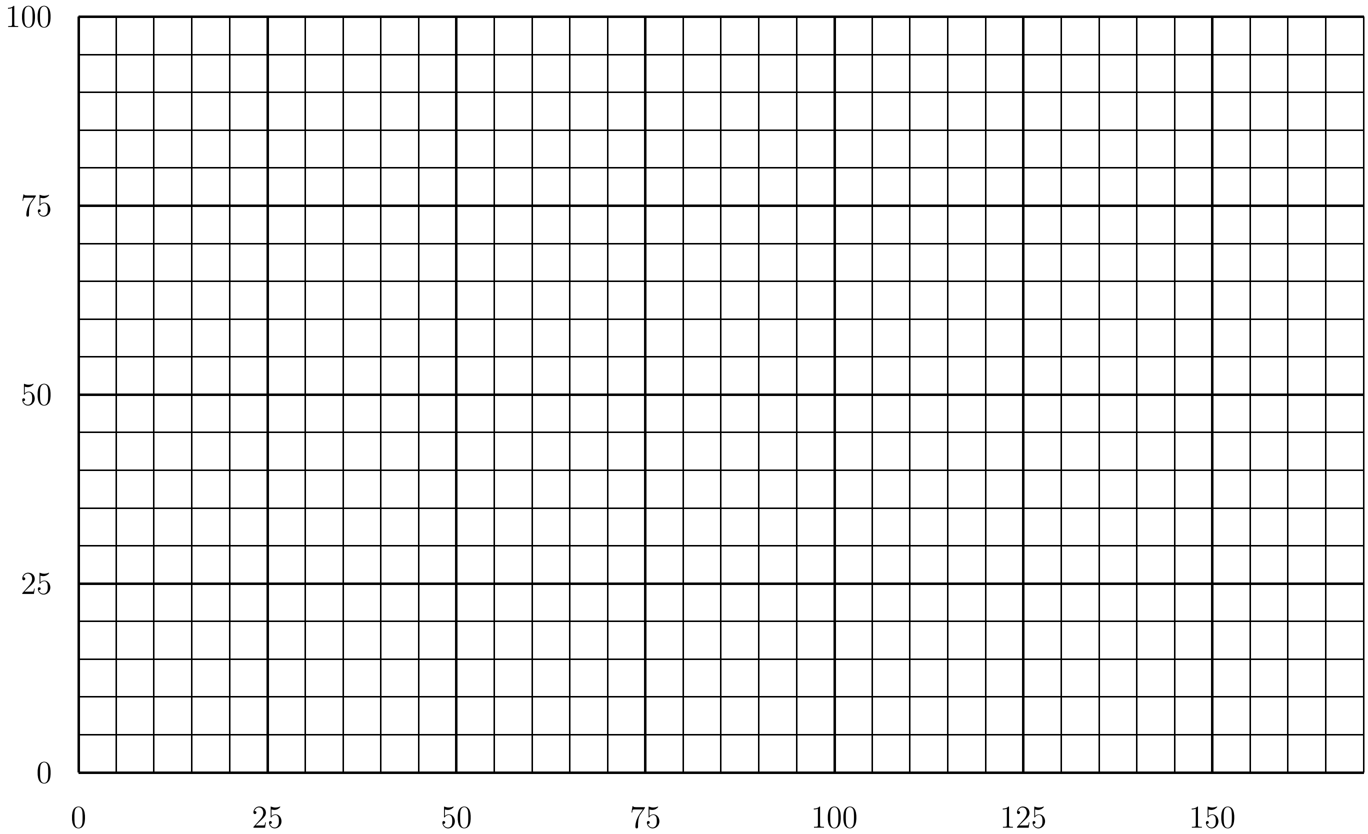
2. *(FS of a discrete-time signal)* Consider the following discrete-time periodic signal

$$x[n] = 2 + 2\cos\left(\frac{\pi}{3}n - \frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{3}n - \frac{\pi}{6}\right)$$

[2] (a) What are the fundamental frequency and fundamental period of $x[n]$?

[5] (b) Find the Fourier series coefficients for $x[n]$.

[3] (c) Plot the magnitude and phase spectrum of $x[n]$ over two periods.

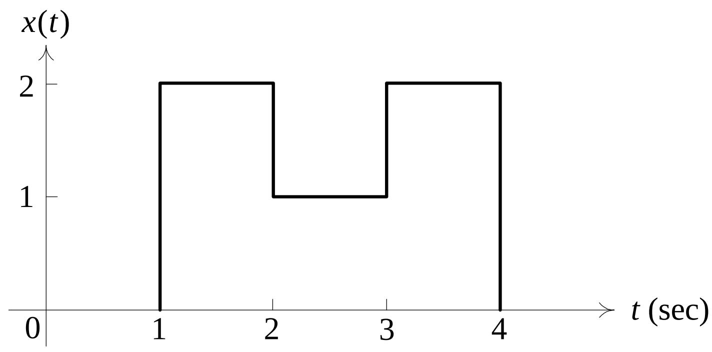


3. *(Fourier Transform)* The signal $x(t)$ has Fourier transform $X(j\omega)$.

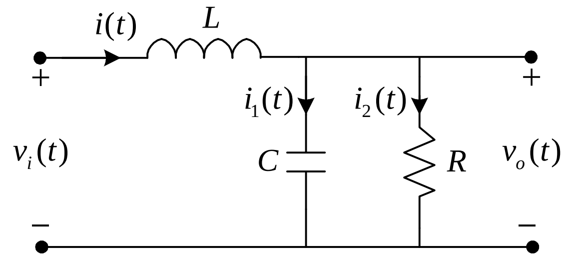
[2] (a) Show that the Fourier transform of $\mathrm{d}x(t)/\mathrm{d}t$ is $(j\omega)X(j\omega)$.

[2] (b) Show that the Fourier transform of $x(t-t_0)$ is $\mathrm{e}^{-j\omega t_0}X(j\omega)$, where t_0 is a constant.

[6] (c) Find $X(j\omega)$ if $x(t)$ is given below.



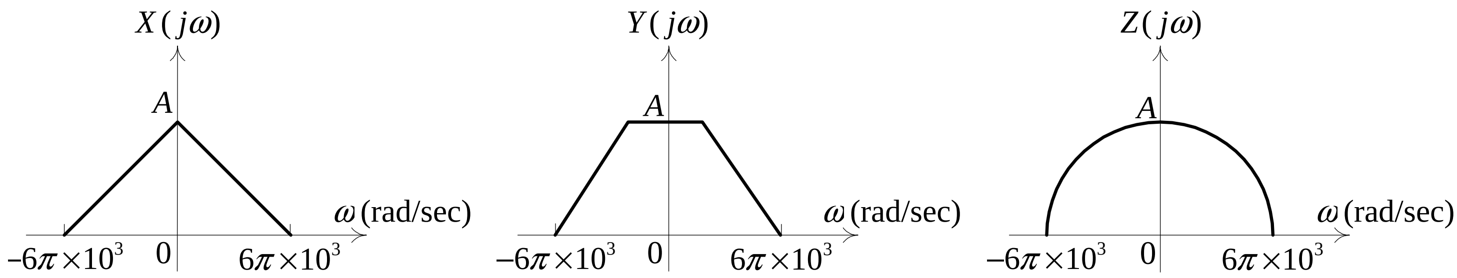
4. *(Frequency response of an RLC filter)* Consider the following circuit:



[5] (a) Show that the frequency response of the circuit is given by $H(j\omega) = \dfrac{1}{1 - \omega^2 LC + j\omega L/R}$

[5] (b) Let $\omega_c = \sqrt{\frac{1}{LC}}$ and $L = 2R^2C$. Show that the magnitude frequency response of the circuit can be written as $|H(j\omega)| = \dfrac{1}{\sqrt{1 + (\omega/\omega_c)^4}}$. Is the circuit a low-pass or high-pass filter? Explain your answer.

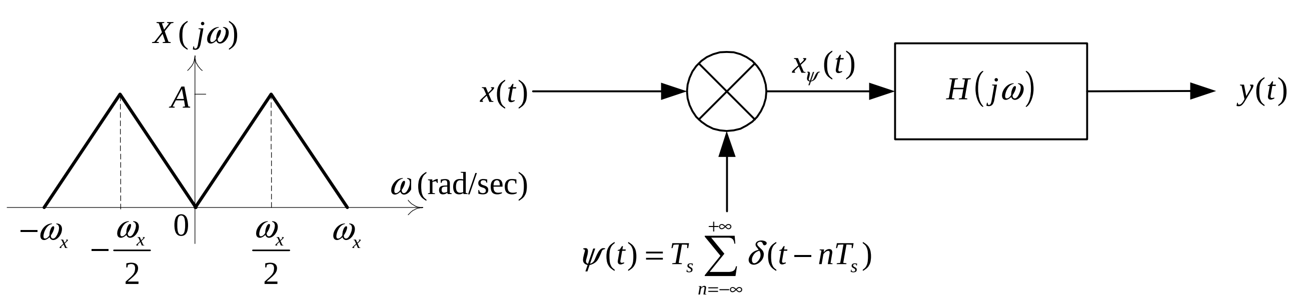
5. *(Amplitude modulation)* The manager of your division asked you to design a communication system to transmit three signals $x(t)$, $y(t)$ and $z(t)$ simultaneously over a radio channel. The spectra of three signals are shown below. You bought a radio frequency spectrum from 650 kHz to 668 kHz (i.e., from $2\pi \times 650 \times 10^3$ rad/sec to $2\pi \times 668 \times 10^3$ rad/sec) and decided to use amplitude modulation.



[5] (a) Clearly present a block diagram of the transmitter to your manager. Also show him the spectrum of the transmitted signal. Clearly identify all the relevant frequencies.

[5] (b) To convince your manager that your design is valid, show and explain to him a block diagram of the receiver that can perfectly recover the signal $y(t)$ (assume that there is no distortion introduced by the channel).

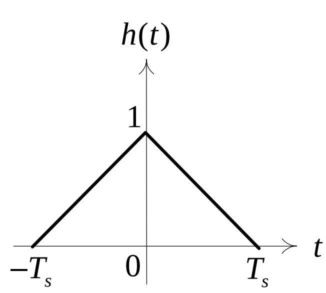
6. *(Sampling)* A block diagram of impulse sampling is shown below.



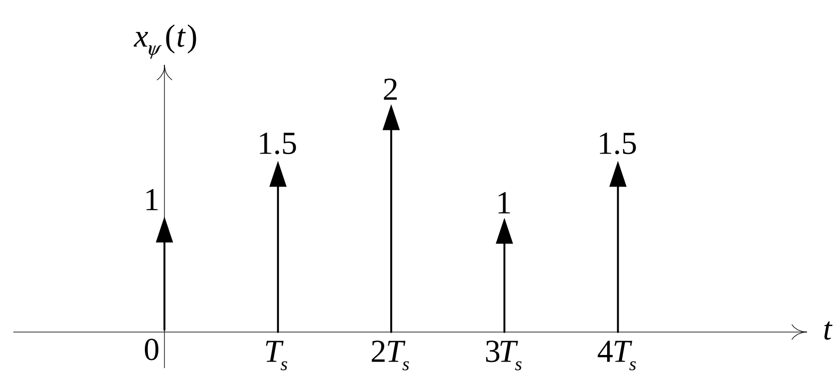
[2] (a) If the sampling frequency is $\omega_s = \frac{2\pi}{T_s} = \frac{3\omega_x}{2}$, does aliasing occur? What is the minimum sampling frequency to prevent aliasing?

[5] (b) Draw the spectrum of $x_\psi(t)$ when $\omega_s = \frac{3\omega_x}{2}$. Also draw the spectrum of $y(t)$ if $H(j\omega)$ is an ideal low-pass filter with a cutoff frequency of $\omega_c = \omega_x$.

[3] (c) Instead of the ideal low-pass filter, consider the filter in Figure (a) as a reconstruction filter. Find and plot the reconstructed signal $y(t)$ if $x_\psi(t)$ is as shown in Figure (b) over the interval $[0, 4T_s]$.



(a)



(b)

Potentially Useful Facts:

- FS representation of DT periodic signals (DTFS):

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \quad a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

- FS representation of CT periodic signals (CTFS):

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} \mathrm{d}t$$

- FT of CT signals:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} \mathrm{d}\omega, \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \mathrm{d}t$$

- Fourier transform pairs/properties:

$$\begin{aligned}
\cos(\omega_0 t) &\xleftrightarrow{\mathcal{FT}} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \\
\delta(t) &\xleftrightarrow{\mathcal{FT}} 1 \\
x(t - t_0) &\xleftrightarrow{\mathcal{FT}} e^{-j\omega t_0} X(j\omega) \\
\frac{\mathrm{d}}{\mathrm{d}t} x(t) &\xleftrightarrow{\mathcal{FT}} j\omega X(j\omega) \\
\int_{-\infty}^{t} x(\tau) \mathrm{d}\tau &\xleftrightarrow{\mathcal{FT}} \frac{1}{j\omega} X(j\omega) \quad \text{(Ignoring DC signal)} \\
\cos(\omega_0 t) x(t) &\xleftrightarrow{\mathcal{FT}} \frac{1}{2} X(j(\omega - \omega_0)) + \frac{1}{2} X(j(\omega + \omega_0)) \\
x(t)\psi(t) &\xleftrightarrow{\mathcal{FT}} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \quad \text{(Impulse sampling)}
\end{aligned}$$

- Identities:

$$\begin{aligned}
e^{j\omega n} &= \cos(\omega n) + j \sin(\omega n) \\
e^{-j\omega n} &= \cos(\omega n) - j \sin(\omega n) \\
\cos(x)\cos(y) &= \frac{1}{2}[\cos(x + y) + \cos(x - y)]
\end{aligned}$$

For my own survey, please indicate ($\surd$) which equations provided above are useful for you, i.e., the ones that you used and you might not remember if not provided. Thanks.